STOCHASTIC DYNAMICAL LOGISTIC
POPULATION GROWTH MODEL

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Abstract

In this paper, we introduce stochastic logistic population growth model. The existence, uniqueness, and asymptotic stability of the solution is discussed. The nonlinear stochastic differential equation of this model is solved numerically. We apply Iran population data in period 1921-2006 for more illustration. The result shows that the approximate solutions have a good degree of accuracy.

1. Introduction

Population growth models are abstract representation of the real world objects, systems or processes to illustrate the theoretical concepts that in these days are increasingly being used in more applied situations such as predicting future outcomes or simulation experimentation. In mathematical literature, many population models have been considered, from deterministic and stochastic population models, where the population size is represented by a discrete random variable, to very complex continuous stochastic models. A non-random case, ignore natural

variation and produce a single value result, while a stochastic model incorporates some natural variation in to model to state unpredictable situations such as weather or random fluctuations in resources and will generate a mean or most probable result. Nowadays, the well-known model like logistic play a major role in modern ecological theory.

In paper [1], the Laguerre-type derivatives and the Laguerre-type exponentials are introduced and then by using of Laguerre-type exponentials, the *L*-exponential and *L*-logistic population growth models are derived, and output of these models is given for world population growth in the period 1955-2005.

The paper [4], develops a stochastic logistic population growth model with immigration and multiple births. The differential equation for the low-order cumulant functions (i.e., mean, variance, and skewness) of the single birth model are reviewed, and the corresponding equations for the multiple birth model are derived. Accurate approximate solutions for the cumulant functions are obtained by using moment closure methods for two families of model parameterizations, one for badger and the other for fox population growth. For both model families, the equilibrium size distribution may be approximated well using the normal approximation, and even more accurately using the saddle point approximation and it is shown that in comparison with the corresponding single birth model, the multiple birth mechanism increases the skewness and the variance of the equilibrium distribution, but slightly reduces its mean. Moreover, the type of density-dependent population control is shown to influence the sign of the skewness and the size of the variance.

The paper [3], discusses the existence, uniqueness, and asymptotic stability of the solution to the stochastic population model with the Allee effect and is focused on the correlation between the growth rate r and population size $N(t)$, which is shown to be positive if the population size $N(t)$ is above the Allee threshold T, and negative if $N(t) < T$.

In paper [2], the stochastic and generalized stochastic exponential population growth models are introduced and the expectations and variances of solutions are obtained.

So, in the present paper; after this introduction, Section 2 presents some preliminaries of stochastic calculus. In Section 3, the deterministic

logistic population growth model is defined. Thereafter, Section 4 introduces the stochastic population growth model. As a case study in Section 5, we consider the population growth of Iran and obtain the output of models for this data and predict the population individuals along 1921-2006 years. Finally, Section 6 gives some brief conclusion.

2. Preliminaries

Definition 2.1 (Brownian motion process)**.** Stochastic process $\{B(t);$ $t \geq 0\}$ is called *Brownian motion process*, if it satisfies the following properties:

(i) $B(0) = 0$.

(ii) $B(t)$ is stationary.

(iii) $B(t)$ has independent increments.

(iv) $B(t) \sim N(0, t)$.

Definition 2.2. Let $\{N(t)\}_{t \geq 0}$ be an increasing family of σ-algebras of sub-sets of Ω. A process $g(t, \omega)$ from $[0, \infty) \times \Omega$ to R^n is called $N(t)$-*adapted* if for each $t \geq 0$, the function $\omega \to g(t, \omega)$ is $N(t)$-measurable, [5, p.25].

Definition 2.3. Let $\nu = \nu(S, T)$ be the class of functions $f(t, \omega)$: $[0, \infty) \times \Omega \to R$ such that,

(i) $(t, \omega) \to f(t, \omega)$, is $B \times \mathcal{F}$-measurable, where B denotes the Borel σ-algebra on $[0, \infty)$ and $\mathcal{F}$ is the σ-algebra on Ω.

(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $B(s); \ s \leq t$.

(iii) $E\big[\int_S^T f^2(t, \omega)dt\big] < \infty$.

Proof. See [5, p.29].

Definition 2.4 [(The Itô integral), [5, p.29]]. Let $f \in v(S, T)$, then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t, \omega) dB(t)(\omega) = \lim_{n \to \infty} \int_S^T \phi_n(t, \omega) dB(t)(\omega), \quad \text{(limit in } L^2(P)),$$

where ϕ_n is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t, \omega) - \phi_n(t, \omega))^2 dt\right] \to 0, \quad \text{as } n \to \infty.$$

Theorem 2.5 (The Itô isometry). *Let* $f \in v(S, T)$, *then*

$$E\left[\left(\int_S^T f(t, w) dB(t)(\omega)\right)^2\right] = E\left[\int_S^T f^2(t, \omega) dt\right].$$

Proof. See [5, p.29].

Theorem 2.6. *Let* $f, g \in v(0, T)$ *and* $0 \le S < U < T$, *then*

(i) $\displaystyle\int_S^T f dB(t) = \int_S^U f dB(t) + \int_U^T f dB(t), \quad$ *for all* ω.

(ii) $\displaystyle\int_S^T (cf + g) dB(t) = c\int_S^T f dB(t) + \int_S^T g dB(t), \quad$ *for all* ω.

(iii) $\displaystyle E\left[\int_S^T f dB(t)\right] = 0.$

(iv) $\displaystyle\int_S^T f dB(t)$ *is* $\mathcal{F}_t$ *-measurable.*

This clearly holds for all elementary functions, so by taking limits, we obtain this for all $f, g \in u(0, T)$.

Proof. See [5, p.30].

Definition 2.7 [(one-dimensional Itô processes), [5, p.43]]. Let $B(t)$ be one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A one-dimensional Itô process (stochastic integral) is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, P)$ of the form

$$X(t) = X(0) + \int_0^t u(s, \omega)ds + \int_0^t v(s, \omega)dB(s),$$

or

$$dX(t) = udt + vdB(t), \tag{1}$$

where

$$P\left[\int_0^t v^2(s, \omega)ds < \infty, \text{ for all } t \geq 0\right] = 1,$$

$$P\left[\int_0^t |u(s, \omega)|ds < \infty, \text{ for all } t \geq 0\right] = 1.$$

Theorem 2.8 (The one-dimensional Itô formula)**.** *Let $X(t)$ be an Itô process given by* (1) *and* $g(t, x) \in C^2([0, \infty) \times R)$, *then*

$$Y(t) = g(t, X(t)),$$

is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2, \tag{2}$$

where $(dX(t))^2 = (dX(t))(dX(t))$ *is computed according to the rules*

$$dt.dt = dt.dB(t) = dB(t).dt = 0, \quad dB(t).dB(t) = dt. \tag{3}$$

Proof. See [5, p.44].

3. Deterministic Logistic Growth Model

In this case, we assume that $a(t) = r(t)$ is an accurate and non-random growth rate at time t. Consider the following population growth model:

$$\frac{dN(t)}{dt} = a(t)N(t)\left(1 - \frac{N(t)}{K}\right), \quad t \geq 0, \tag{4}$$

where $N(t)$ is the number of population individuals at time t, $N(0)$ is the non-random initial number at time $t = 0$, K is the environmental

carrying capacity, which represents the maximal population size, and $\frac{N(t)}{K}$ is the environmental resistance. We can see that the solution of (4) is as follows:

$$N(t) = \frac{KN(0)}{N(0) + (K - N(0))e^{-\int_0^t r(t)\,dt}}. \tag{5}$$

By assumption $r(t) = r$ (r is a constant value), we get

$$N(t) = \frac{KN(0)}{N(0) + (K - N(0))e^{-rt}}. \tag{6}$$

4. Stochastic Logistic Growth Model

In stochastic forms, $a(t)$ at time t is not completely definite and it depends on some random environment effects, i.e.,

$$a(t) = r(t) + \text{``}noise\text{''},$$

where $r(t)$ is a non-random function of time variable that means the growth rate of population at time t, whereas we do not know the exact behaviour of "*noise*" term, we can set

$$a(t) = r(t) + \alpha(t)W(t),$$

where $W(t) = \frac{dB(t)}{dt}$ is one-dimensional white noise process and $B(t)$ is a one-dimensional Brownian motion and $\alpha(t)$ is a non-random function that shows the infirmity and intensity of noise at time t. Now, in the stochastic logistic model,

$$\frac{dN(t)}{dt} = N(t)\left(1 - \frac{N(t)}{K}\right)\left(r(t) + \alpha(t)\frac{dB(t)}{dt}\right), \quad t \geq 0,$$

we can rewrite it as

$$dN(t) = N(t)\left(1 - \frac{N(t)}{K}\right)\left(r(t)dt + \alpha(t)dB(t)\right), \quad t \geq 0, \tag{7}$$

where $B = \{B(t),\, t \geq 0\}$ is a one-dimensional standard Brownian motion defined on a probability space $\{\Omega, F, \{F_t\}_{t\geq 0}, P\}$ with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (it is right continuous and increasing, while F_0 contains all P-null sets), $N(0)$ is a random variable independent of B such that $0 < N(0) < K$ a.s., and $N(t)$ is an unknown stochastic process, that is, the solution to Equation (7) satisfying the initial condition $N(0)$. Because of the logical requirement that $N(t)$ must be positive, we can not directly apply the usual existence-and-uniqueness theorem. That is why we need a procedure, which is explained in detail in the next subsection.

4.1. Existence and uniqueness of the positive solution

It is obvious that $N(t) = 0$ and $N(t) = K$ are the solutions to Equation (7), so, suppose that $N(t) \neq 0$ and $N(t) \neq K$. By applying the Itô formula, we observe that

$$d\ln\left|\frac{K - N(t)}{N(t)}\right| = d\ln|K - N(t)| - d\ln|N(t)|$$

$$= -\frac{dN(t)}{K - N(t)} - \frac{(dN(t))^2}{2(K - N(t))^2} - \frac{dN(t)}{N(t)} + \frac{(dN(t))^2}{2N^2(t)}$$

$$= -\left(\left(r(t) - \frac{\alpha^2(t)}{2} + \frac{\alpha^2(t)}{2}\left(\frac{N(t)}{K}\right)^2\right.\right.$$

$$\left.\left. + \alpha^2(t)\frac{N(t)}{K}\right)dt + \alpha(t)\,dB(t)\right).$$

Then,

$$\frac{K - N(t)}{N(t)} = C\exp[A(t)], \tag{8}$$

where

$$C = \frac{K - N(0)}{N(0)} \quad \text{and} \quad A(t) = -\int_0^t \left(r(s) - \frac{1}{2}\alpha^2(s) + \alpha^2(s)\frac{N(s)}{K}\right)ds + \alpha(s)dB(s).$$

In dependence on the initial population size $N(0)$, we distinguish two cases:

(i) Let $0 < N(0) < K$. Then $C = \dfrac{K - N(0)}{N(0)} > 0$ a.s., so that

$$0 < N(t) < K, \quad t \geq 0. \tag{9}$$

Hence, we conclude from (8) that

$$N(t) = \frac{KN(0)}{N(0) + (K - N(0))\exp[A(t)]}. \tag{10}$$

(ii) Let $0 < K < N(0)$. Then $C = \dfrac{K - N(0)}{N(0)} < 0$ a.s., so that

$$0 < K < N(t), \quad t \geq 0. \tag{11}$$

Hence, we conclude from (8) that

$$N(t) = \frac{KN(0)}{N(0) - (K - N(0))\exp[A(t)]}. \tag{12}$$

In special case, if $r(t) = r$ and $\alpha(t) = \alpha$ (r, α are constant values), we get

$$N(t) = \frac{KN(0)}{N(0) \pm (K - N(0))\exp[\Psi(t)]}, \tag{13}$$

where

$$\Psi(t) = -\left(\left(rt - \frac{\alpha^2}{2}t + \frac{\alpha^2}{K}\int_0^t N(s)\right)ds + \alpha B(t)\right),$$

and if $\alpha = 0$, we have (6).

In what follows, we formulate the existence-and-uniqueness theorem of the positive solution $N(t)$ to Equation (7). Moreover, we prove the uniform continuity of $N(t)$ in the sense that it is continuous and its almost every sample path is uniformly continuous on $t \geq 0$. In order to prove the uniform continuity of the positive solution, we apply the Kolmogorov-Centsov theorem on the continuity of a stochastic process derived from the moment property [3].

Theorem 4.1. *For any initial value $N(0)$ such that $0 < N(0) < K$, there exists a unique, uniformly continuous positive solution to Equation (7).*

Proof. Let $0 < N(0) < K$. In view of (9), we define a stochastic process $\{x(t),\, t \geq 0\}$ by

$$x(t) := \ln \frac{K - N(t)}{N(t)}.$$

If we apply the Itô formula for $x(t)$, we find that

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad t \geq 0, \tag{14}$$

where

$$f(x(t)) = -(r(t) + \alpha^2(t)\frac{(1 - e^x)}{2(1 + e^x)}),$$

and

$$g(x(t)) = -\alpha(t),$$

for all $t > 0$ and $x(0) := \ln \dfrac{K - N(0)}{N(0)}$. It is easy to see that the functions f and g are bounded and continuous, as well as that they satisfy the local Lipschitz condition and the linear growth condition. Hence, Equation (12) has a unique continuous solution $x(t),\, t \geq 0$ satisfying the initial condition $x(0)$. Since

$$N(t) = \frac{K}{1 + e^{x(t)}},$$

we will prove that $N(t)$ is the solution to Equation (7). Indeed,

$$dN(t) = d[\frac{K}{1 + e^{x(t)}}] = \frac{-Ke^{x(t)}}{(1 + e^{x(t)})^2}[dx(t) + \frac{1 - e^{x(t)}}{2(1 + e^{x(t)})}dx(t)dx(t)]$$

$$= \frac{K}{1 + e^{x(t)}} \frac{e^{x(t)}}{1 + e^{x(t)}} (r(t)dt + \alpha(t)dB(t))$$

$$= N(t)\left(1 - \left(\frac{N(t)}{K}\right)\right)(r(t)dt + \alpha(t)dB(t)). \qquad \square$$

Now, we assume that $r(t) = r$ and $\alpha(t) = \alpha$. In order to prove that almost every sample path of $N(t)$ is uniformly continuous for $t \geq 0$, we will consider Equation (7) in its integral form, that is,

$$N(t) = N(0) + \int_0^t f(N(s))ds + \int_0^t g(N(s))dB(s), \quad t \geq 0,$$

where $0 < N(0) < K$, and

$$f(N(s)) = rN(s)\left(1 - \left(\frac{N(s)}{K}\right)\right),$$

$$g(N(s)) = \alpha N(s)\left(1 - \left(\frac{N(s)}{K}\right)\right).$$

Let $0 < v < \nu < \infty, \nu - v \leq 1$, and $p > 2$. By using the Hölder inequality and the well-known moment inequality for the Itô integrals, we get

$$E|N(t) - N(s)|^p \leq 2^{p-1}(\nu - v)^{p-1} \int_0^t E[f(N(s))]^p ds$$

$$+ 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(\nu - v)^{\frac{p}{2}-1} \int_v^\nu E[g(N(s))]^p ds. \quad (15)$$

Since,

$$E|f(N(s))|^p \leq (rK)^p,$$

and

$$E|g(N(s))|^p \leq (\alpha K)^p,$$

we find from (13) that

$$E|N(t) - N(s)|^p \leq A(\nu - v)^{\frac{p}{2}},$$

where

$$A = 2^{p-1} K^p \left(r^p(s) + \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \alpha^2(s) \right).$$

The application of the Kolmogorov-Centsov theorem on the continuity of a stochastic process ([3]), implies that almost every sample path of $N(t)$ is locally but uniformly Hölder-continuous with exponent $\gamma \in (0, \frac{p-2}{2p})$ and, therefore, uniformly continuous on $t \geq 0$.

As usual, we prove the uniqueness of the positive solution by contradiction. Suppose that $N_1(t)$ and $N_2(t)$ are two positive solutions of Equation (7) with the same initial value $N(0)$, where $0 < N(0) < K$. Then,

$$d(N_2(t) - N_1(t)) = (N_2(t) - N_1(t))\left(1 - \frac{N_2(t) - N_1(t)}{K}\right)(rdt + \alpha dB(t)).$$

Let us denote that $J(t) = N_2(t) - N_1(t)$. Then,

$$J(t) = \int_0^t J(s)\left(1 - \frac{N_2(s) - N_1(s)}{K}\right)(rds + \alpha dB(s)).$$

By using the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$. Hölder inequality, Itô isometry, and the fact that $0 < N_i(t) < K, i = 1, 2,$ it follows from (9)

$$E(J(t))^2 \leq 2(r^2 t + \alpha^2)E\int_0^t J^2(s)\left(1 - \frac{N_2(s) - N_1(s)}{K}\right)^2 ds$$

$$\leq 2(r^2 t + \alpha^2)E\int_0^t J^2(s)ds.$$

Finally, the application of the Gronwall-Bellman lemma yields that $E(J(t))^2 = 0$, and therefore,

$$E(N_2(t) - N_1(t))^2 = 0.$$

According to the Chebyshev inequality, we find for an arbitrary $\epsilon > 0$;

$$P\{|N_2(t) - N_1(t)| \geq \epsilon\} \leq \frac{1}{\epsilon^2} E|N_2(t) - N_1(t)|^2 = 0.$$

Hence, $N_1(t) = N_2(t)$ a.s. for all $t \geq 0$. Thus, the proof becomes complete.

4.2. Stability of the positive solution

Since Equation (7) is not explicitly solvable, it is important to investigate the behaviour of the positive solution in long period of time. To do this, we will apply several times the following elementary inequality:

$$\frac{1}{a} \leq \frac{\ln a - \ln b}{a - b} \leq \frac{1}{b}, \quad 0 < b < a, \tag{16}$$

as well as the well-known assertion.

Lemma 4.1 ([3]). *Let* $f : [0, \infty] \to [0, \infty]$ *be an integrable and uniformly continuous function. Then,*

$$\lim_{t \to \infty} f(t) = 0.$$

The following theorem describes the asymptotic mean square stability of the positive solution to Equation (7), when the initial population size is $0 < N(0) < K$.

Theorem 4.2. *Let* $N(t)$ *be a uniformly continuous positive solution to Equation* (7) *with the initial value* $N(0)$, *where* $0 < N(0) < K$. *Then,*

(a) *If* $r > \alpha^2$, *then* $\lim_{t \to \infty} E(K - N(t))^2 = 0$.

(b) *If* $r < -\alpha^2$, *then* $\lim_{t \to \infty} E(N(t))^2 = 0$.

Proof. First, it follows from (9) that $0 < N(t) < K$; $t \geq 0$, since $0 < N(0) < K$.

(a) The application of the Itô formula to $V^2(t)$, where

$$V(t) = \ln K - \ln N(t), \quad t \geq 0 \tag{17}$$

is the Lyapunov function, implies that

$$dV^2(t) = 2V(t)dV(t) + (dV(t))^2$$

$$= -2(\ln K - \ln N(t))\left(\frac{K - N(t)}{K}\right)[(r - \frac{(K - N(t))}{2K}\alpha^2)dt + \alpha \, dB(t)]$$

$$+ \frac{(K - N(t))^2}{K^2}\alpha^2 dt,$$

or,

$$EV^2(t) = EV^2(0) + E\int_0^t[-2(\ln K - \ln(N(s)))\frac{(K - N(s))}{K}(r - \frac{(K - N(s))}{2K}\alpha^2)$$

$$+ \frac{(K - N(t))^2}{K^2}\alpha^2]ds. \tag{18}$$

By applying the inequality (14) and the assumption $r > \alpha^2$, we get

$$\frac{dEV^2(t)}{dt} < -\frac{2}{K}E[\frac{(K - N(t))^2}{N(t)}(r - \frac{(K - N(t))}{2K}\alpha^2) - \frac{(K - N(t))^2}{2K}\alpha^2)]$$

$$= -\frac{2}{K^2}E[(K - N(t))^2(r - \frac{(K - N(t))}{2K}\alpha^2 - \frac{\alpha^2}{2})]$$

$$< -R \, E[(K - N(t))^2],$$

where $R = \frac{2}{K^2}(r - \alpha^2)$ is a positive constant. Thus, $EV^2(t)$ is decreasing and hence,

$$EV^2(t) < EV^2(0) - R\int_0^t E(K - N(s))^2 ds.$$

Since,

$$EV^2(0) = E(\ln K - \ln(N(0)))^2 < \infty,$$

then

$$EV^2(0) + R\int_0^t E(K - N(s))^2 ds < EV^2(0) < \infty,$$

which implies that $E(K - N(t))^2 \in L^1[0, \infty]$. The application of Theorem 4.1 yields that $E(K - N(t))^2$ is uniformly continuous on $[0, \infty]$ and Lemma 4.1 that

$$\lim_{t \to \infty} E(K - N(t))^2 = 0.$$

(b) Analogously to the first part of the proof, if we apply the Itô formula to $V^2(t)$, where

$$V(t) = \ln(K) - \ln(K - N(t)), \quad t \geq 0 \tag{19}$$

is the Lyapunov function, we see that

$$dV^2(t) = -2(\ln(K) - \ln(K - N(t)))\frac{N(t)}{K}[(-r - \frac{N(t)}{2K}\alpha^2)\,dt$$

$$+ \alpha\,\frac{N(t)}{K}\,dB(t)] + \frac{N^2(t)}{K^2}\alpha^2 dt.$$

Similarly to the previous discussion,

$$\frac{dEV^2(t)}{dt} < -D\,E(N(t))^2,$$

where $D = -\dfrac{2}{K^2}(r + \alpha^2)$ is a constant, and hence $EV^2(t)$ is decreasing. So,

$$EV^2(t) + D\int_0^t E(N(s)^2)\,ds < EV^2(0) < \infty,$$

where $EV^2(0) = E(\ln K - \ln(K - N(0)))^2 < \infty$. Therefore, $E(N(t))^2 \in L^1[0, \infty]$. According to Theorem 4.1 and Lemma 4.1, it follows that

$$\lim_{t \to \infty} E(N(t))^2 = 0,$$

which completes the proof.

Conditions under which the positive solution to Equation (7) is asymptotically stable in mean are given in the next theorem. Note that the greater interval is obtained for r, than the one from Theorem 4.1.

Theorem 4.3. *Let $N(t)$ be a uniformly continuous positive solution to Equation* (7) *with the initial value $N(0)$, where $0 < N(0) < K$. Then,*

(a) *If $r > \dfrac{\alpha^2}{2}$, then $\lim_{t \to \infty} EN(t) = K$.*

(b) *If $r < -\dfrac{\alpha^2}{2}$, then $\lim_{t \to \infty} EN(t) = 0$.*

Proof. As above, we have $0 < N(t) < K$; $t \geq 0$.

(a) To prove the first part, we use the Lyapunov function

$$V(t) = \ln(K) - \ln(N(t)), \quad t \geq 0,$$

and apply the Itô formula

$$dV(t) = -\frac{dN(t)}{N(t)} + \frac{dN(t)^2}{2N(t)^2} = -\frac{K - N(t)}{K}(rdt + \alpha \, dB(t)) + \frac{\alpha^2}{2K^2}(K - N(t))^2 dt$$

$$= -\frac{K - N(t)}{K}\left[(r - \frac{(K - N(t))}{2K}\alpha^2)dt + \alpha \, dB(t)\right]. \tag{20}$$

Then,

$$EV(t) = EV(0) - E\int_0^t \frac{K - N(s)}{K}\left(r - \frac{(K - N(s))}{2K}\alpha^2\right)ds.$$

Since $r > \dfrac{\alpha^2}{2}$, then

$$\frac{dEV(t)}{dt} < -\frac{1}{2K}(2r - \alpha^2)E(K - N(t)) = -PE(K - N(t)),$$

where $P = \dfrac{1}{2K}(2r - \alpha^2)$ is a positive constant. Likewise, since $EV(t)$ is decreasing and $EV(0) = E(\ln(K) - \ln(N(0))) < \infty$, then

$$EV(t) + P\int_0^t E(K - N(s))ds < EV(0) < \infty,$$

so that $E(K - N(t)) \in L^1[0, \infty]$, which leads to the conclusion that

$$\lim_{t\to\infty} E(K - N(t)) = 0.$$

(b) To prove the second part, we use the fact that $r < -\dfrac{\alpha^2}{2}$ and the Lyapunov function $V(t)$ given by (17), i.e., $V(t) = \ln(K) - \ln(K - N(t))$, $t \geq 0$. Then, apply the Itô formula

$$dV(t) = \frac{dN(t)}{K - N(t)} + \frac{(dN(t))^2}{2(K - N(t))^2} = \frac{N(t)}{K}(rdt + \alpha dB(t)) + \frac{(N(t))^2}{2K^2}\alpha^2 dt.$$

By repeating completely the previous procedure, we see that

$$EV(t) = EV(0) - E\int_0^t \frac{N(s)}{K}\left(-r - \frac{N(s)}{2K}\alpha^2\right)ds,$$

so,

$$\frac{dEV(t)}{dt} < -E(N(t))\frac{1}{K}\left(-r - \frac{\alpha^2}{2}\right) = -R E(N(t)),$$

where $R = -\dfrac{1}{2K}(2r + \alpha^2) > 0$ is a constant. By omitting details, we finally conclude that

$$\lim_{t\to\infty} E(N(t)) = 0.$$

We can prove the above theorems for $0 < K < N(t)$ or $0 < K < N(t)$ similarly.

The previous theorem shows that under certain conditions, the deterministic logistic population model (4) and the corresponding stochastic logistic differential equation (7) have a similar property-the global stability of the positive and bounded solutions.

5. Numerical Solution

In this section, we solve nonlinear equation (10) for $K = 150000000$, $r(t) = t\sqrt{\dfrac{N(t)}{N(0)}} - 1$ for each census period, and $\alpha = 0.05$, for Iran population in the period 1921-2006 [2] numerically.

We see the results in the table below:

Table 1. Exact population and predicted values of Iran population in the period 1921-2006

Year	Population	Predicted population	Relative error
1921	9707000	9707000	0
1926	10456000	10123825	0.013746
1931	11196292	11063000	0.011905
1936	11981523	12544000	0.046945
1941	12858145	13931000	0.083438
1946	14199617	12872000	0.093497
1951	16309190	16067000	0.014850
1956	19077139	18529000	0.028733
1966	26065751	26129000	0.002427
1976	34190685	31248000	0.086067
1986	50472079	46841000	0.071942
1991	57081935	58416000	0.023371
1996	61411110	60759000	0.010619
2001	72157981	68493000	0.050791
2006	75354709	72785000	0.034102
Mean	------	------	0.03816

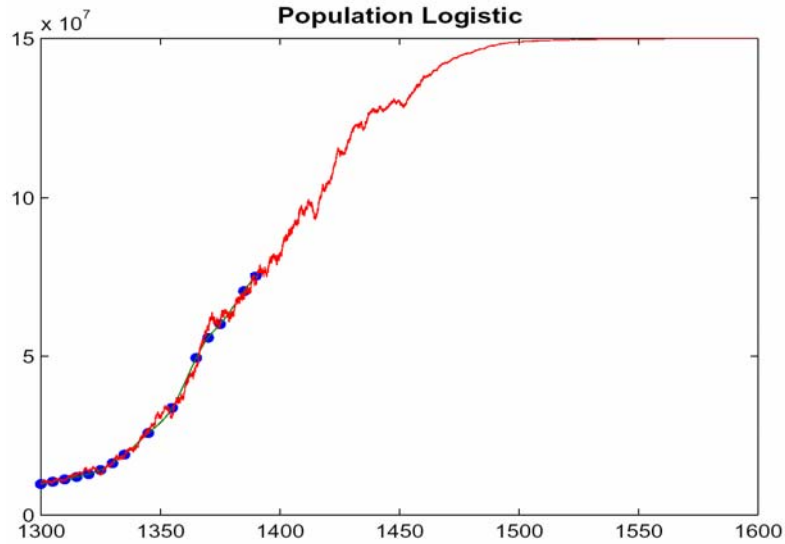The comparison plot is shown in Figure 1.

Figure 1. The solid red line represents predicted population by our method and the blue dot line represents exact population.

6. Conclusion

The modelling of systems by deterministic differential equations usually requires the parameters that are completely known. But in some cases, their values may depend on the microscopic properties of the medium in a complicated way and may fluctuate due to some external or internal random "noise". Here, we defined stochastic logistic population growth model, where the so-called parameter, population growth rate is not completely definite and it depends on some random environmental effects. Then, we discussed the existence, uniqueness, and asymptotic stability of the solution. Because it is impossible to find the exact solution of nonlinear stochastic differential equation of (10), we solved it numerically.

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